Statistical Models For Data In Compact Groups

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Content

- **What is group scale?**
  . . . , thaught in an example

Content

- **What is group scale?**
- **Classical instances of group scale.**
  Directions, axis, orientations, . . .

Content

- **What is group scale?**
- **Classical instances of group scale.**
- **What is so special about group scale?**
  What is not special?

Content

- What is group scale?
- Classical instances of group scale.
- What is so special about group scale?
- Selected methods

Content

- What is group scale?
- Classical instances of group scale.
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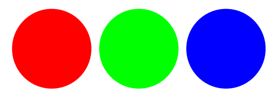
- Selected methods
- Outlook

Content

- **What is group scale?**
- **Classical instances of group scale.**
- **What is so special about group scale?**
- **Selected methods**
- **Outlook**
- **Conclusions**

What is group scale?

A statistical problem

We have 3 workers: 

and 3 shifts

D	D	N

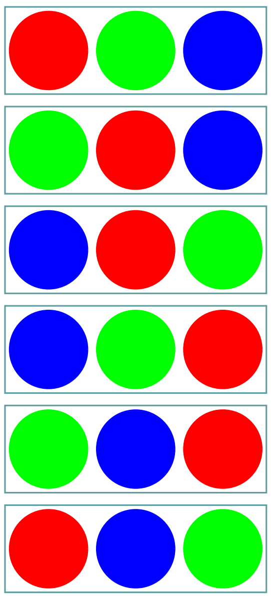
D is day shift

N is night shift

What is group scale?

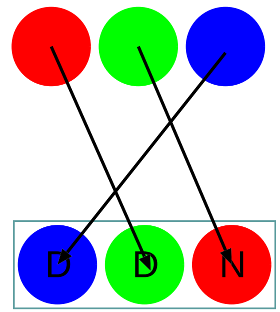
A statistical problem

possible outcomes:



Each week the boss assigns the workers to shifts:

We have 3 workers:



and 3 shifts

D is day shift

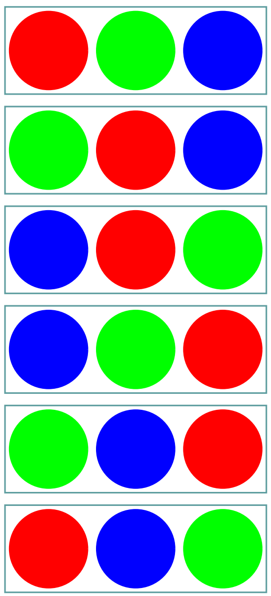
N is night shift

What is group scale?

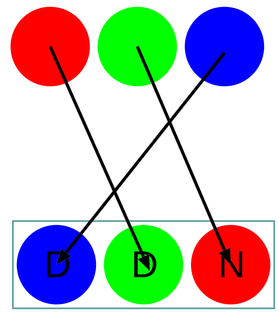
A statistical problem

Each week the boss assigns the workers to shifts:

possible outcomes:



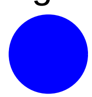
We have 3 workers:



and 3 shifts

D is day shift

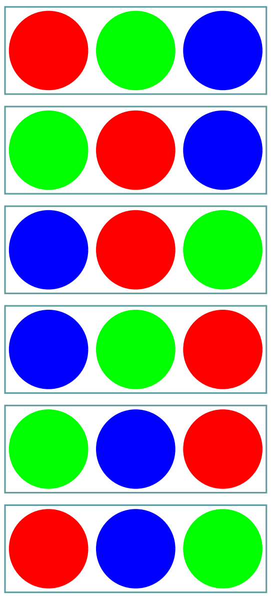
N is night shift

Mr.  , a disabled, feels discriminated by beeing unequally treated in the assignment to shifts.

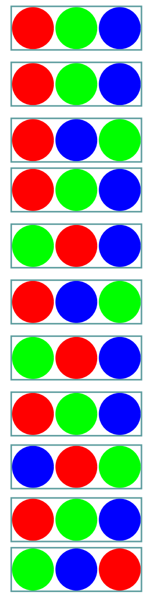
What is group scale?

The statistical data

possible outcomes:



Each week the boss assigns the workers to shifts:



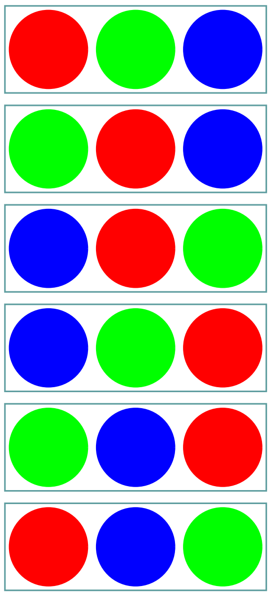
A dataset from the last 11 weeks.

What is a group?

The group S3

Elements:

1=



What is a group?

The group S3

Elements:

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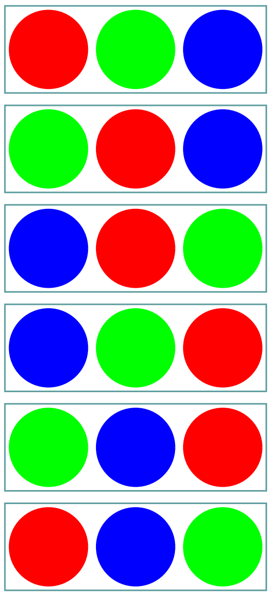
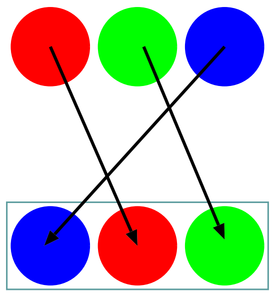


Interpretation of elements:

Startconfigeration:



Element

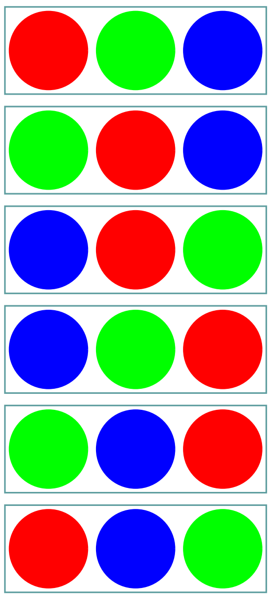
Elements describe a reordering

What is a group?

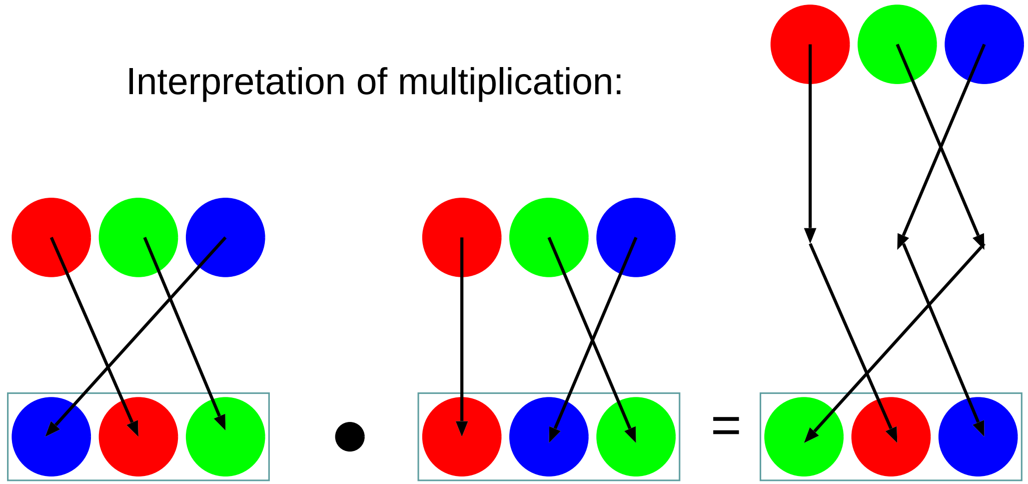
The group S3

Elements:

1=



Interpretation of multiplication:



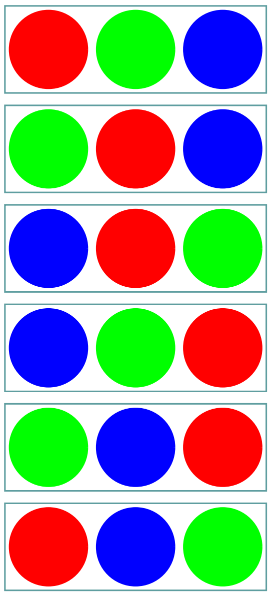
● is sequential application of reordering

What is a group?

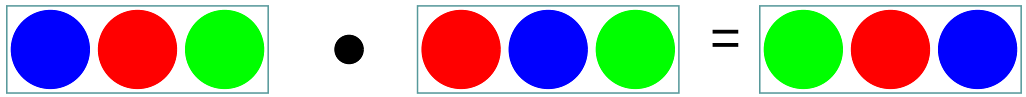
A group is:

A set of Elements:

1=



with a multiplication:



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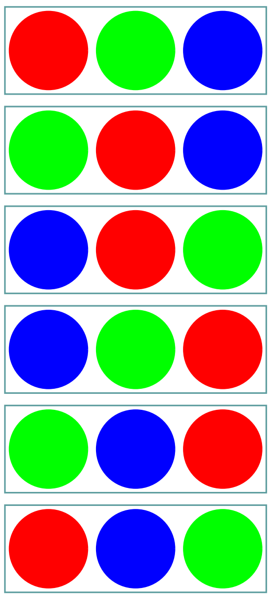

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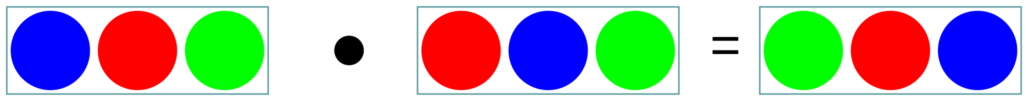
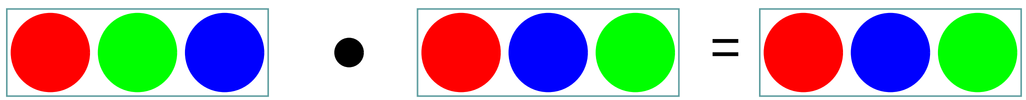


having a 1, doing nothing:



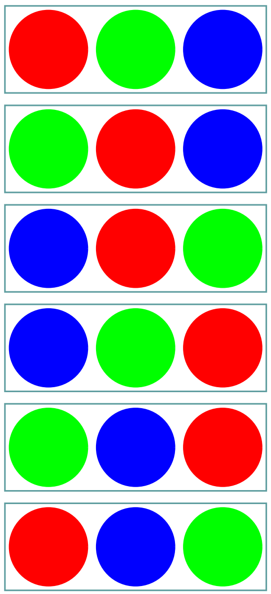
1 a a

What is a group?

A group is:

A set of Elements:

1=



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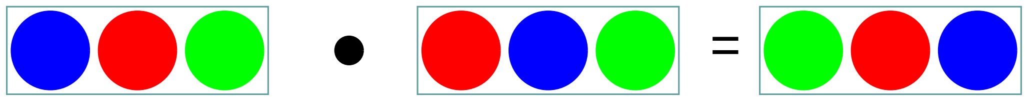
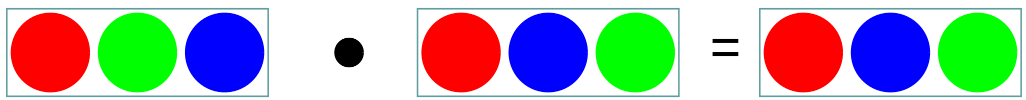
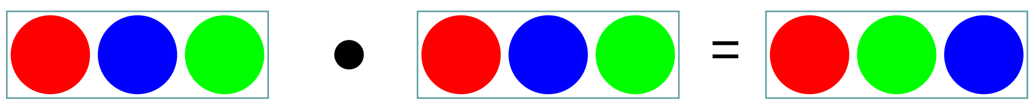


having a 1, doing nothing:



1 a a

and inverse elements, inverting the reordering:



a^{-1} a 1

Motivation of equivalence classes

The following predications might or might not be true:

- The two day shifts are similar/indistinguishable.

Motivation of equivalence classes

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- Mr. Blue has the same qualification as Mr. Green.

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The following predications might or might not be true:

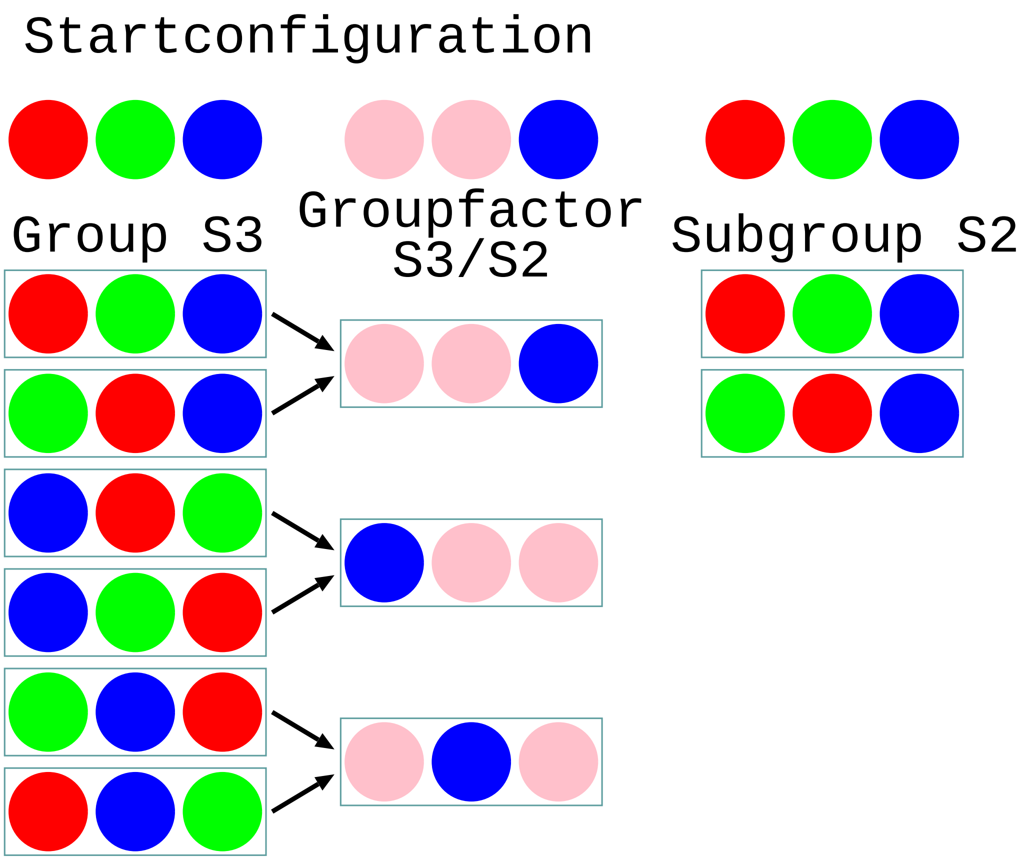
- The two day shifts are similar/indistinguishable.
- Mr. Blue has the same qualification as Mr. Green.
- Mr. Red and Mr. Green are similar.

Motivation of equivalence classes

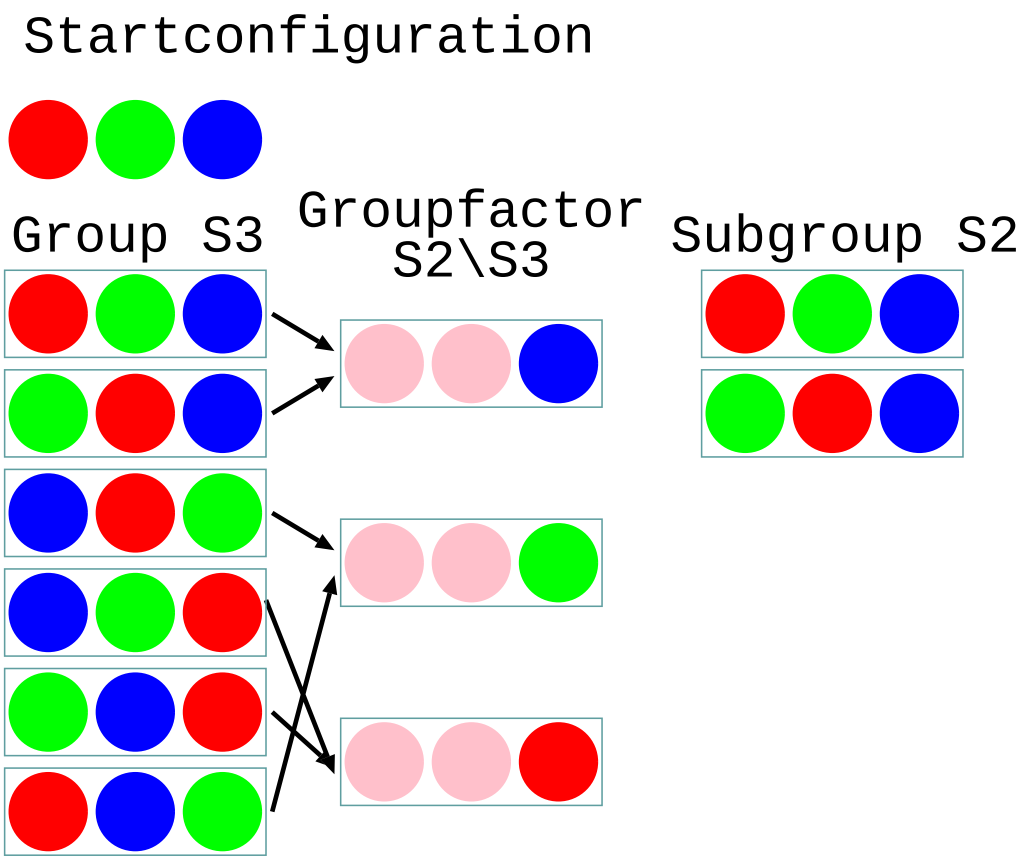
The following predications might or might not be true:

- The two day shifts are similar/indistinguishable.
- Mr. Blue has the same qualification as Mr. Green.
- Mr. Red and Mr. Green are similar.
- The assignment of Mr. Red and Mr. Green to shifts is not available due to privacy reasons.

Right equivalence classes



Left equivalence classes



Types of equivalence classes

We can distinguish 3 types of equivalence classes:

Right group quotients:

$$G/S := \{gS : g \in G\}$$

Left group quotient:

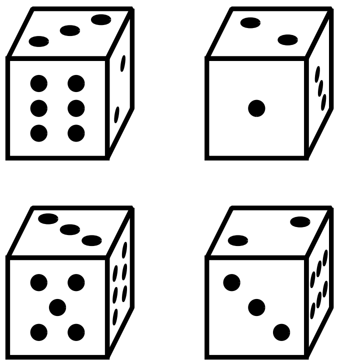
$$S\backslash G := \{Sg : g \in G\}$$

Double group quotients:

$$S\backslash G/T := \{SgT : g \in G\}$$

Examples of group scale

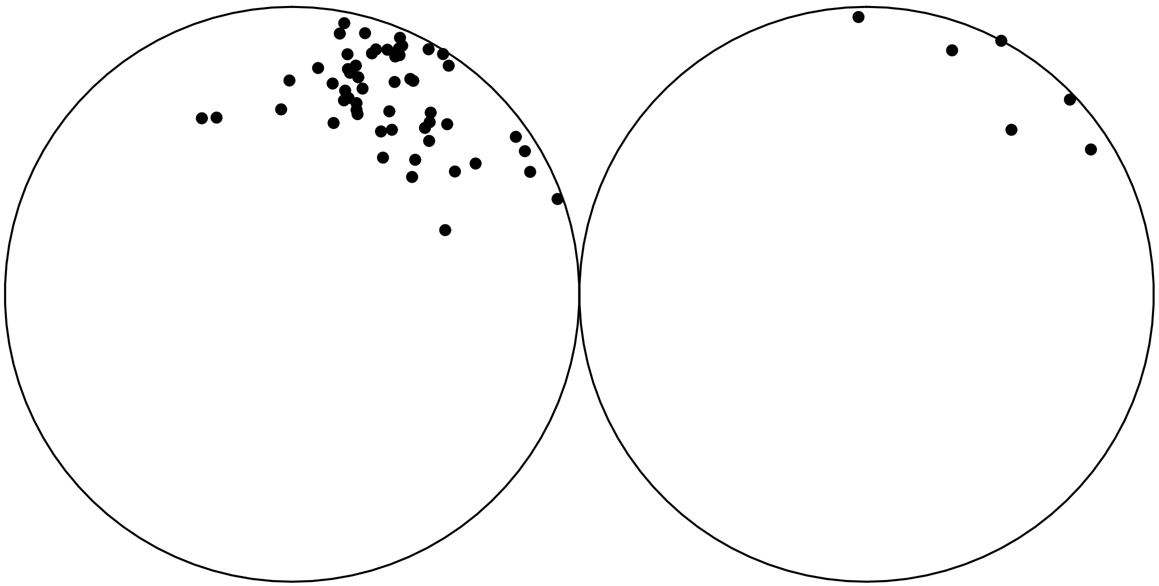
- **Object Orientations**



Crystallography, Industrial processes, . . .

Examples of group scale

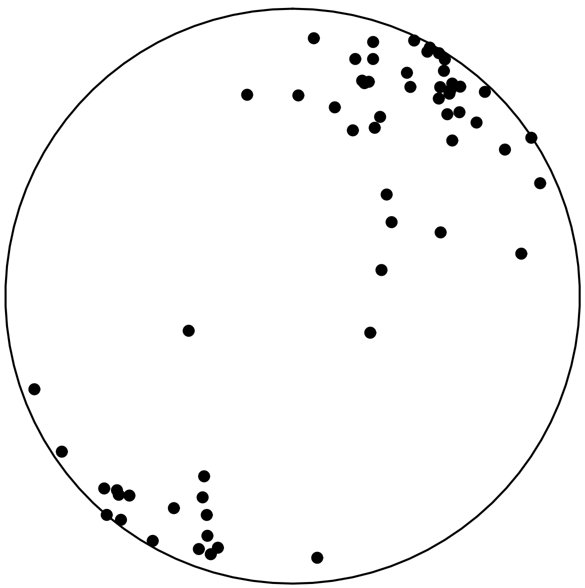
- **Object Orientations**
- **Spherical/Directional data**



paleomagnetic orientation,
normals of sedimentations plains,
orientation of line dislocations, pole wandering
Books: Fisher/Lewis/Embleton (1987), Mardia (1972), Watson (1983)

Examples of group scale

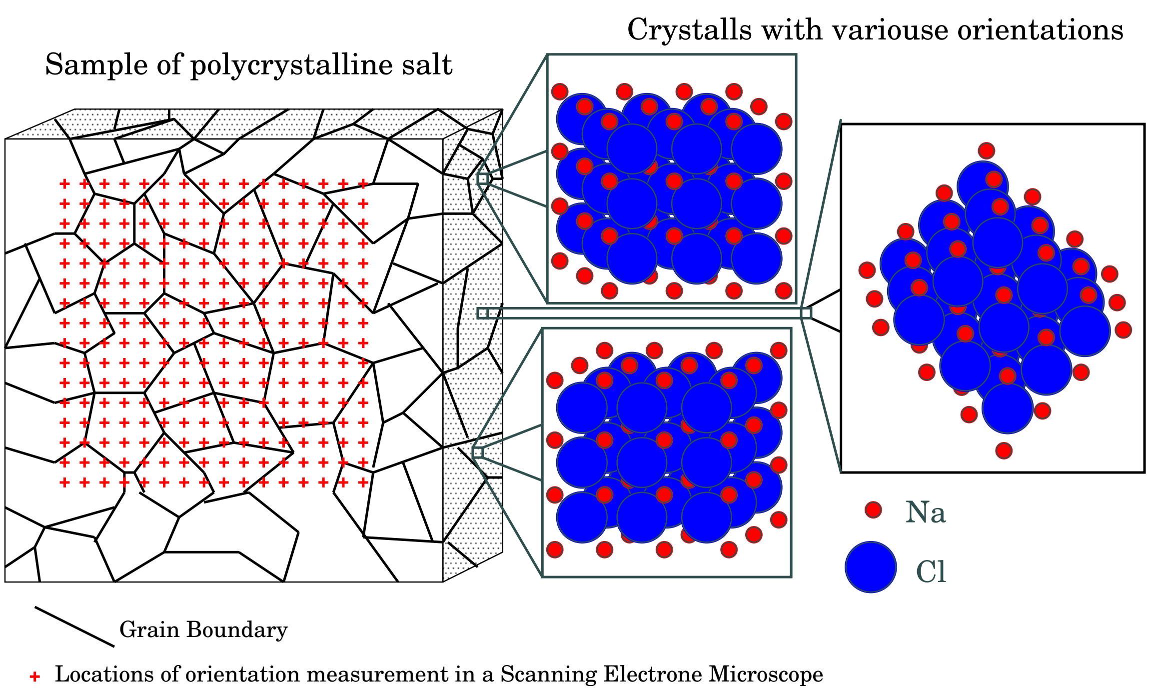
- **Object Orientations**
- **Spherical/Directional data**
- **Axial data**



Normals of schist plains, lineation of rocks, crystallographic c-axis
fiber orientation of polymeres, magnetic susceptibilty

Examples of group scale

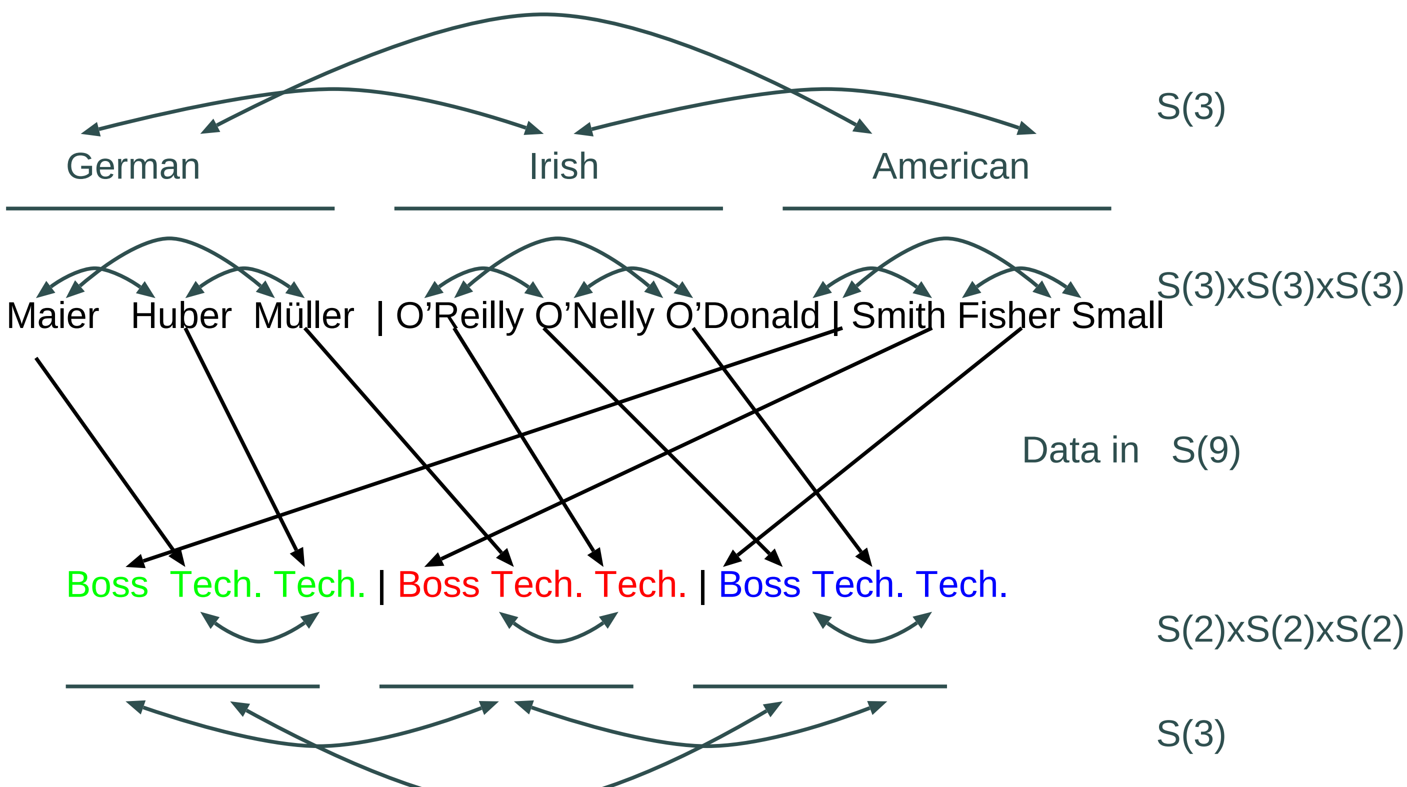
- **Object Orientations**
- **Spherical/Directional data**
- **Axial data**
- **Crystallographic orientations**



Used in geology, material sciences, chip production, alluminum industry

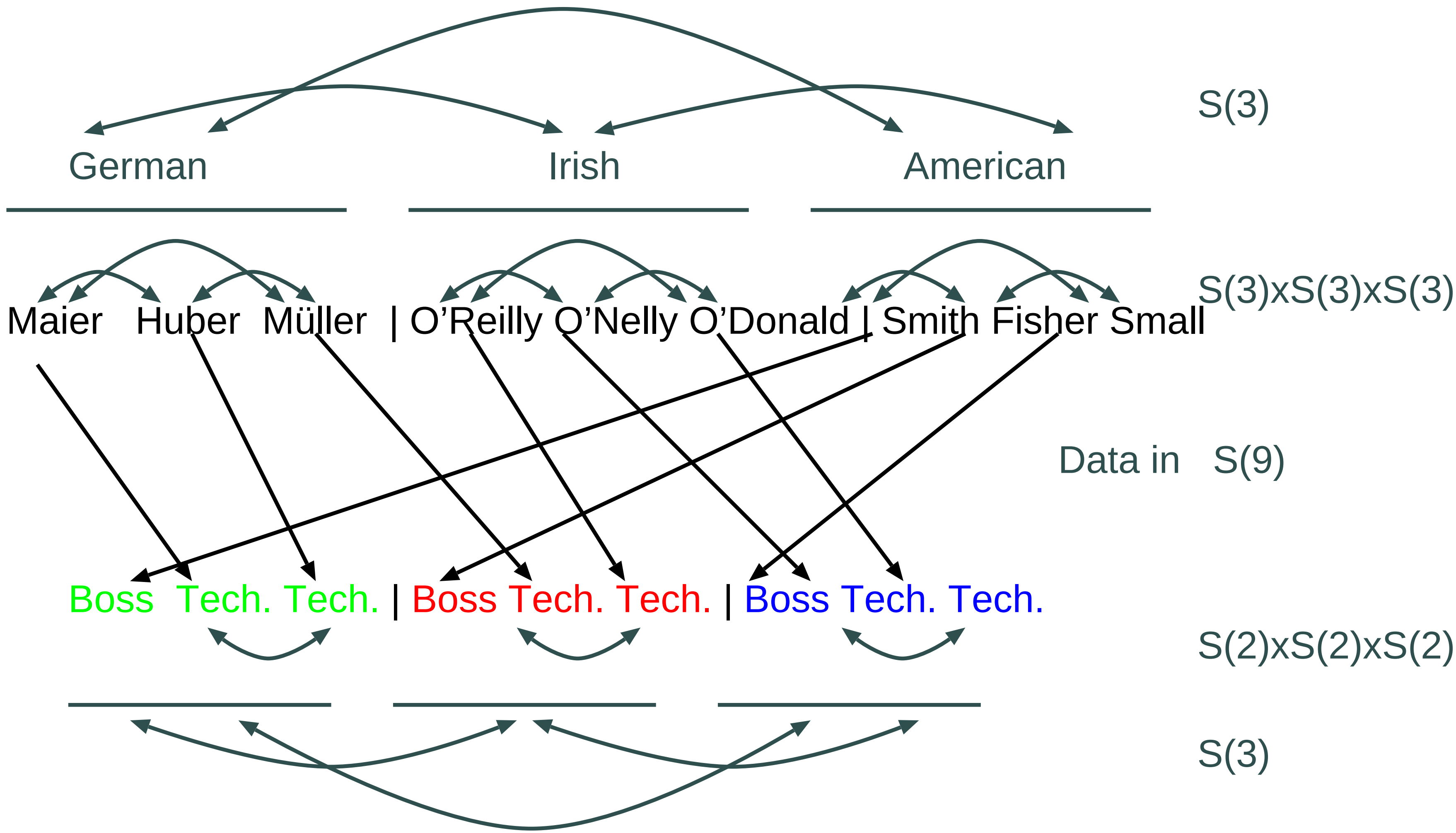
Examples of group scale

- **Object Orientations**
- **Spherical/Directional data**
- **Axial data**
- **Crystallographic orientations**
- **Assignments**



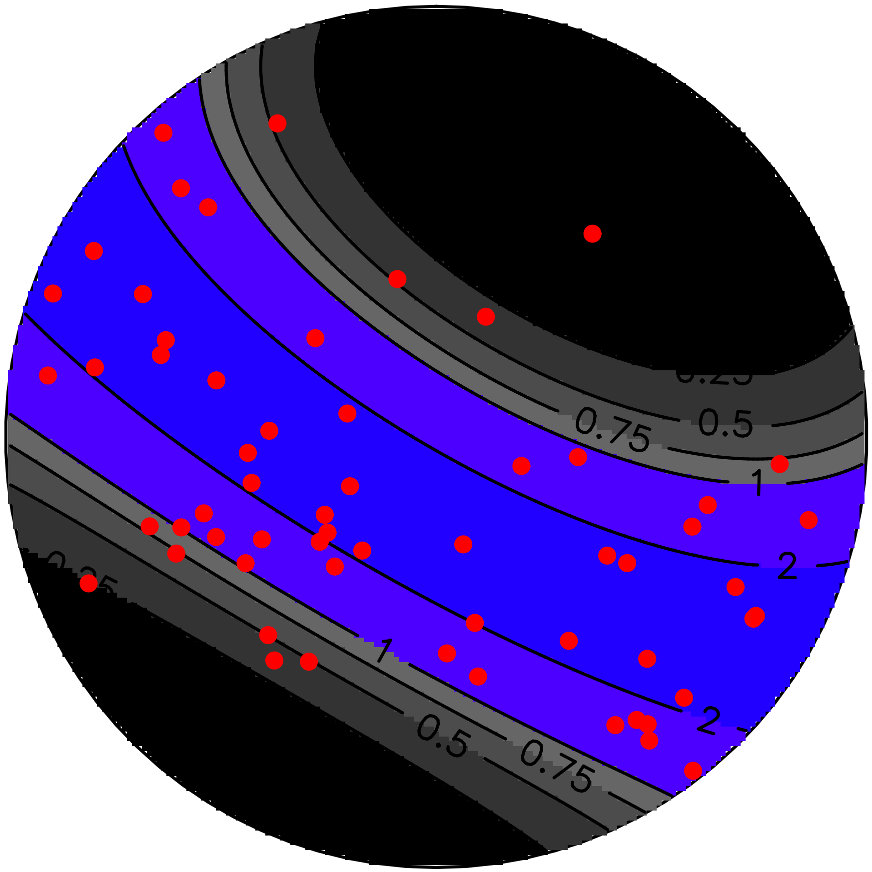
Examples of symmetry

- **Exchangebility of objects and groups**



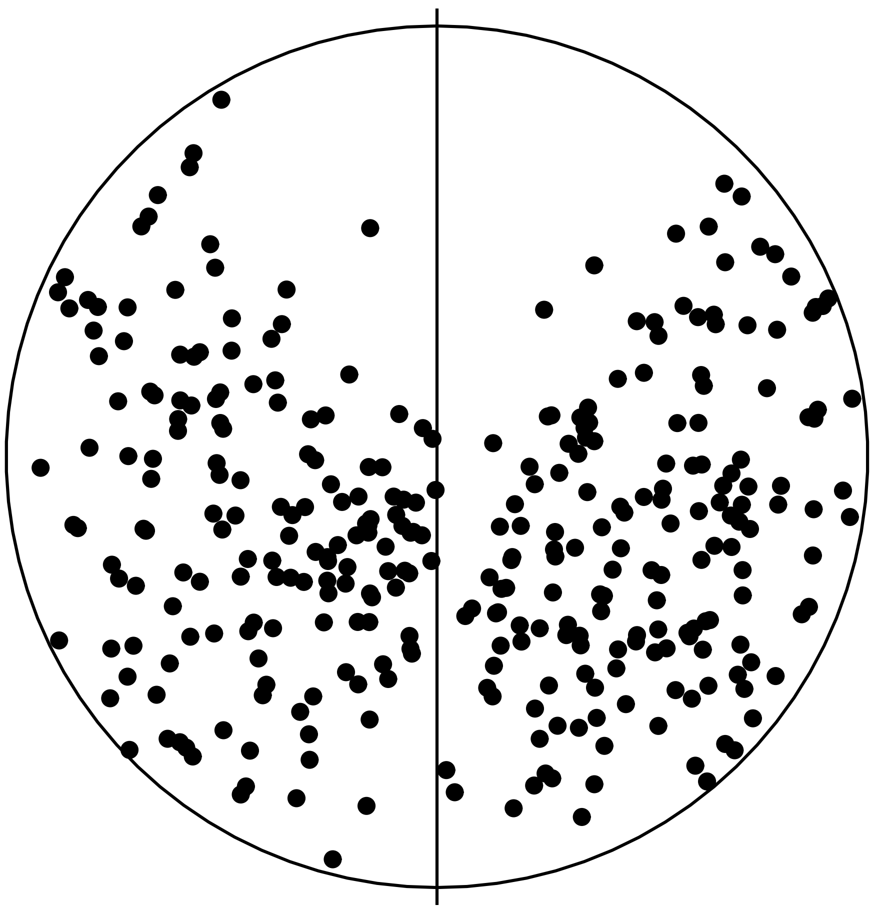
Examples of symmetry

- **Exchangebility of objects and groups**
- **Axial Rotation symmetry**



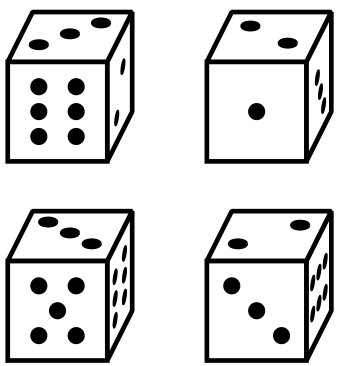
Examples of symmetry

- **Exchangebility of objects and groups**
- **Axial Rotation symmetry**
- **Mirror Symmetry**



Examples of symmetry

- **Exchangebility of objects and groups**
- **Axial Rotation symmetry**
- **Mirror Symmetry**
- **Symmetry of forms and lattices**



What is different in group scale?

$\surd$ **means that a replacement is available in group scale.**

- **No $+$, only a $\cdot$**
  no summing, $\surd$
  no expectation,
  no moments, $\surd$
  no mean, $\surd$
  no variance, $\surd$
  no $y = a + bx + \varepsilon$ $\surd$

What is different in group scale?

$\surd$ **means that a replacement is available in group scale.**

- **No $+$, only a $\cdot$**
- **No 0, only a 1**
  no $H_0 : \mu = 0 vs. H_0 : \mu \neq 0$, $\surd$
  no $E[\varepsilon] = 0$ $\surd$

What is different in group scale?

$\surd$ **means that a replacement is available in group scale.**

- **No** $+$**, only a** $\cdot$
- **No** 0**, only a** 1
- **No** $<$
  no cdf, $\surd$
  no ranks,
  no quantiles,
  no median,
  no histograms

What is different in group scale?

$\surd$ **means that a replacement is available in group scale.**

- **No** $+$**, only a** $\cdot$
- **No** 0**, only a** 1
- **No** $<$
- **No euclidean space**
  no undistorted scatterplot,
  no least squares, $\surd$
  no PCA, $\surd$
  no euclidean/manhattan/Mahalanobis distance, $\surd$
  no normal distribution, $\surd$
  no characteristic function, $\surd$
  no Lesbegue measure $\surd$

What is different in group scale?

$\surd$ **means that a replacement is available in group scale.**

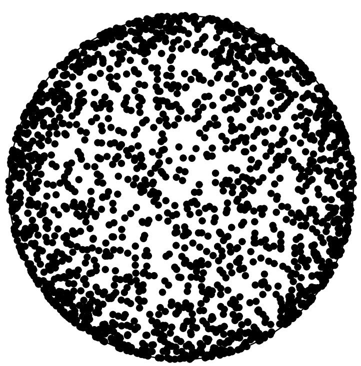
- **No** $+$**, only a** $\cdot$
- **No** 0**, only a** 1
- **No** $<$
- **No euclidean space**
- **No idea**
  $\ldots$, how to work in group scale. $\surd$

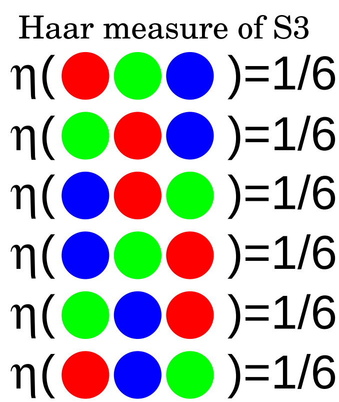
Haar measure and densities

- **Haar measure η**
  For compact groups there is one unique probability measure η with forall $g \in GABARDINE$

$$\eta(A) = \eta(gA)$$

  η_G is the uniform distribution on G.
  $d\eta_G$ replaces $d\boldsymbol{\lambda}$



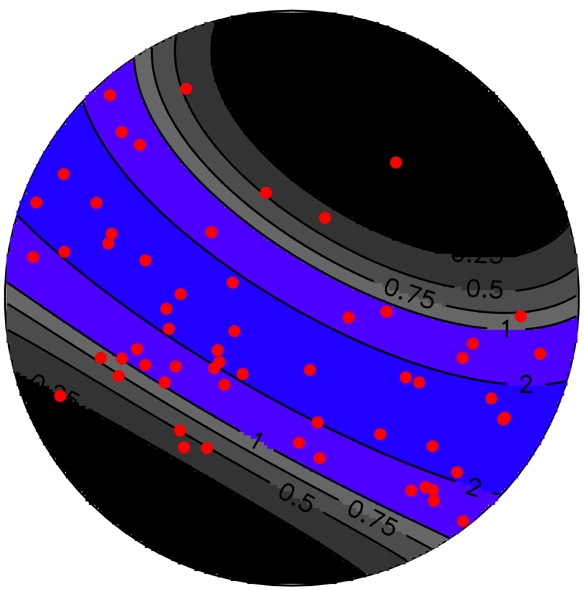


Haar measure and densities

- **Haar measure η**
- **Probability densities**

$$P(A) = \int_A f(g) d\eta_G(g)$$

$f(g) =$ multiples of uniform distribution at g

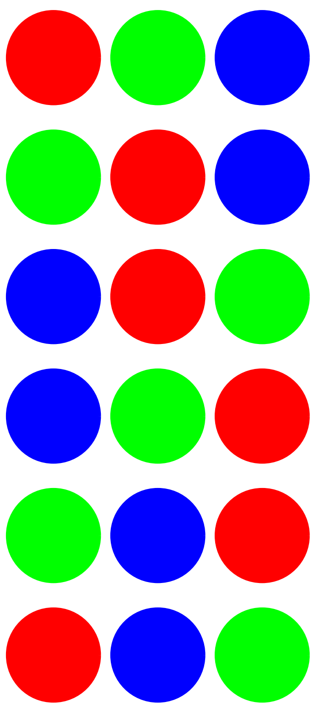


Representations

Compact Groups have a (more or less) unique sequence of characteristic representations

$$T_l : G \longrightarrow \mathbb{R}^{d_l \times d_l}$$

Group S3



$T_0(g)$	$T_1(g)$	$T_2(g)$
(1)	(1)	$\begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}$
(1)	(-1)	$\begin{pmatrix} -\sin 0 & \cos 0 \\ \cos 0 & \sin 0 \end{pmatrix}$
(1)	(1)	$\begin{pmatrix} \cos 120 & -\sin 120 \\ \sin 120 & \cos 120 \end{pmatrix}$
(1)	(-1)	$\begin{pmatrix} -\sin 120 & \cos 120 \\ \cos 120 & \sin 120 \end{pmatrix}$
(1)	(1)	$\begin{pmatrix} \cos 240 & -\sin 240 \\ \sin 240 & \cos 240 \end{pmatrix}$
(1)	(-1)	$\begin{pmatrix} -\sin 240 & \cos 240 \\ \cos 240 & \sin 240 \end{pmatrix}$

Representations

Compact Groups have a (more or less) unique sequence of characteristic representations

$$T_l : G \to \mathbb{R}^{d_l \times d_l}$$

- **Representations are homomorphisms**

$$T(g_1 g_2) = T(g_1)T(g_2)$$

Representations

Compact Groups have a (more or less) unique sequence of characteristic representations

$$T_l : G \to \mathbb{R}^{d_l \times d_l}$$

- **Representations are homomorphisms**
- **Their matrix elements** $T_l^{ij}(g)$ **span** $L^2(\eta_G)$ **as orthogonal Hilbert basis.**

$$T_l(g) = \begin{pmatrix} T_l^{11}(g) & \cdots & T_l^{1d_l}(g) \\ \vdots & \ddots & \vdots \\ T_l^{d_l 1}(g) & \cdots & T_l^{d_l d_l}(g) \end{pmatrix}$$

Representations

Compact Groups have a (more or less) unique sequence of characteristic representations

$$T_l : G \longrightarrow \mathbb{R}^{d_l \times d_l}$$

- **Representations are homomorphisms**
- **Their matrix elements** $T_l^{ij}(g)$ **span** $L^2(\eta_G)$ **as orthogonal Hilbert basis.**
- $\text{span}\left\langle T_l^{ij}(g) \right\rangle$ **are the only invariant subspaces of** $L^2(\eta_G)$**.**
  Only information given in these subspaces is invariant under group operations.

Moments

- **Using characteristic representaitions, we can define (uncentered/matrix valued) moments.**

$$\mu_l := E[T_l(g)]$$

Moments

- **Using characteristic representaitions, we can define (uncentered/matrix valued) moments.**

$$\mu_l := E[T_l(g)]$$

- μ_l **is compatible with group action**

$$E[X+c] = E[X] + c \quad \leftrightarrow \quad E[T_l(\sigma g)] = T_l(\sigma)\mu_l$$

Moments

- **Using characteristic representaitions, we can define (uncentered/matrix valued) moments.**

$$\mu_l := E[T_l(g)]$$

- μ_l **is compatible with group action**
- **With convolutions** μ_l **behaves like a classical mean**

$$E[X+Y] = E[X] + E[Y] \quad \leftrightarrow \quad \mu_l^{P*Q} = \mu_l^P \mu_l^Q$$

Moments

- **Using characteristic representaitions, we can define (uncentered/matrix valued) moments.**

$$\mu_l := E[T_l(g)]$$

- μ_l **is compatible with group action**
- **With convolutions** μ_l **behaves like a classical mean**
- $(\mu_l)_l$ **uniquely defines** P
  . . . like a sequence of all moments and like the characteristic function.

Moments

- **Using characteristic representaitions, we can define (uncentered/matrix valued) moments.**

$$\mu_l := E[T_l(g)]$$

- μ_l **is compatible with group action**
- **With convolutions** μ_l **behaves like a classical mean**
- $(\mu_l)_l$ **uniquely defines** P
- $(\mu_l)_l$ **is a characteristic transform**
  Convolution correspond to products like with characteristic functions:

$$(f * g)(x) = f^*(x)g^*(x) \quad \leftrightarrow \quad (\mu_l^{P*Q})_l = (\mu_l^P \mu_l^Q)_l$$

Moments

- **Using characteristic representaitions, we can define (uncentered/matrix valued) moments.**

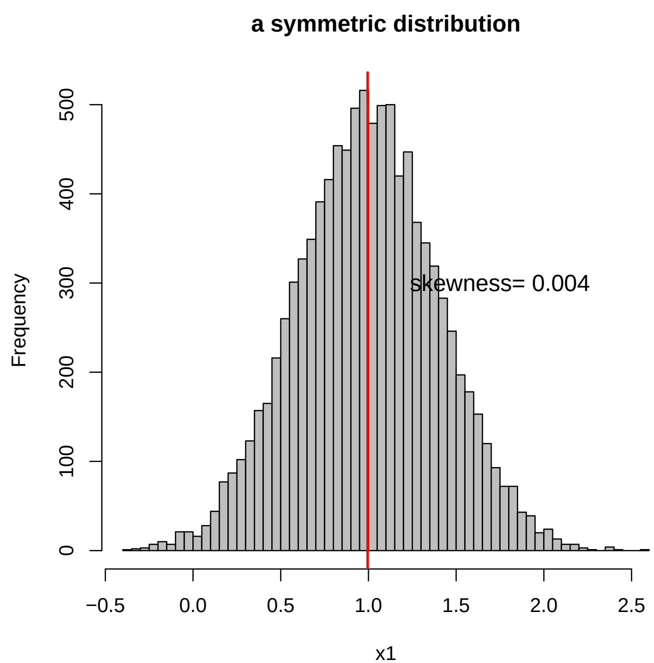
$$\mu_l := E[T_l(g)]$$

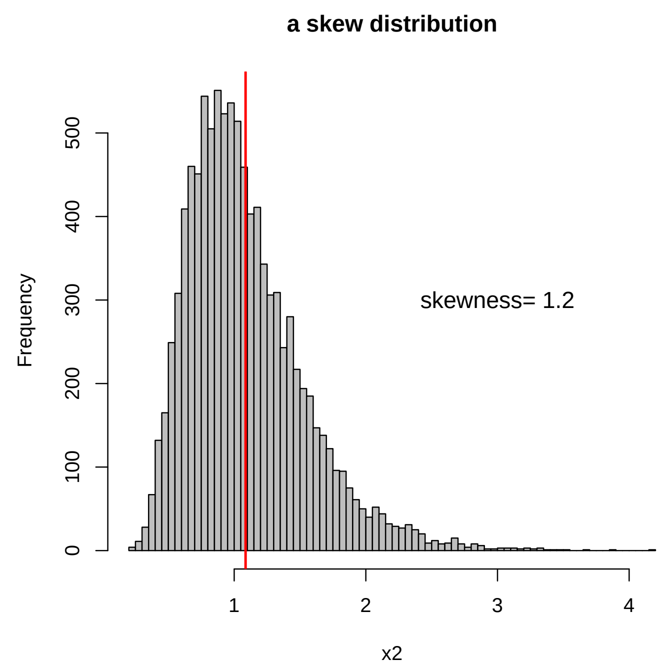
- μ_l **is compatible with group action**
- **With convolutions** μ_l **behaves like a classical mean**
- $(\mu_l)_l$ **uniquely defines** P
- $(\mu_l)_l$ **is a characteristic transform**
- **However** $\mu_1 \notin G$
  μ_1 does not replace the mean as a measure of location

Symmetry and moments

- **Classical example: Skewness**
  Symmetry around the mean implies

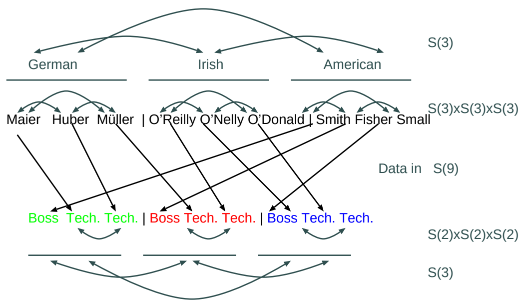
$$E[(x-\mu_1)^3]=0$$

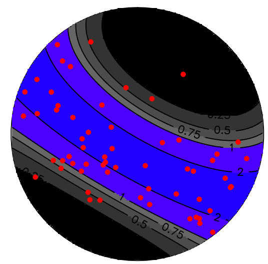


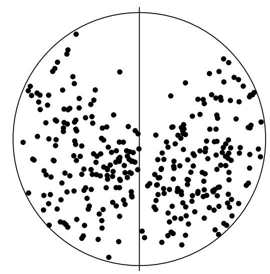


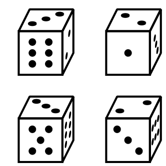
Symmetry and moments

- **Classical example: Skewness**
- **Symmetry in groups scale is subgroup symmetry**









Symmetry and moments

- **Classical example: Skewness**
- **Symmetry in groups scale is subgroup symmetry**
- **Symmetry can be checked by linear conditions on the moments**
  P is (left-)symmetric with respect to a subgroup S if and only if forall l

$$\mathrm{im}\mu_l \perp \mathop{\mathbf{span}}_{s\in S} \mathrm{im}(T_l(s) - T_l(0))$$

Usefull for checking for symmetry:

$$\mathbf{H}_0 : N_l\mu_l = 0 \quad \text{vs.} \quad \mathbf{H}_1 : N_l\mu_l \neq 0$$

with $\mathrm{im}N_l^t = \mathop{\mathbf{span}}_{s\in S} \mathrm{im}(T_l(s) - T_l(0))$

Symmetry and moments

- **Classical example: Skewness**
- **Symmetry in groups scale is subgroup symmetry**
- **Symmetry can be checked by linear conditions on the moments**
- **Symmetric representations** $\overset{\vdots}{T}_l(g)$

  If P is (left-)symmetric with repect to a subgroup S the T_l can be replaced by:

$$\overset{\vdots}{T}_l(g) := A_l T_l(g)$$

  With some orthogonal rectangular matrix A_l spanning $\ker N_l$. Usefull for information reduction when a symmetry is part of the Hypothesis.

Symmetry and moments

- **Classical example: Skewness**
- **Symmetry in groups scale is subgroup symmetry**
- **Symmetry can be checked by linear conditions on the moments**
- **Symmetric representations** $\overset{:}{T}_l(g)$
- **Symmetric moments**

$$\mu_l^{S\cdot} := E[\overset{:}{T}_l(g)] = A_l \mu_l$$

Types of symmetric representations

We can distinguish 3 types of equivalence classes:

Right group quotients G/S:

$$\dot{T}_l(g) := T_l(g)A_l^t$$

Left group quotients $S\backslash G$:

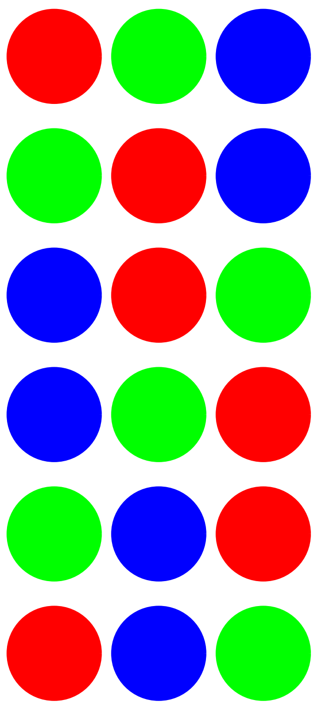
$$\overset{:}{T}_l(g) := A_l T_l(g)$$

Double group quotients $S\backslash G/R$:

$$\overset{\therefore}{T}_l(g) := A_l T_l(g) B_l^t$$

Application to example

- **Exchangebility of nondisabled workers**



Group S3	$T_0(g)$	$T_1(g)$	$T_2(g)$
	(1)	(1)	$\begin{pmatrix}\cos 0 & -\sin 0\\ \sin 0 & \cos 0\end{pmatrix}$
	(1)	(-1)	$\begin{pmatrix}-\sin 0 & \cos 0\\ \cos 0 & \sin 0\end{pmatrix}$
	(1)	(1)	$\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}$
	(1)	(-1)	$\begin{pmatrix}-\sin 120 & \cos 120\\ \cos 120 & \sin 120\end{pmatrix}$
	(1)	(1)	$\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}$
	(1)	(-1)	$\begin{pmatrix}-\sin 240 & \cos 240\\ \cos 240 & \sin 240\end{pmatrix}$

Application to example

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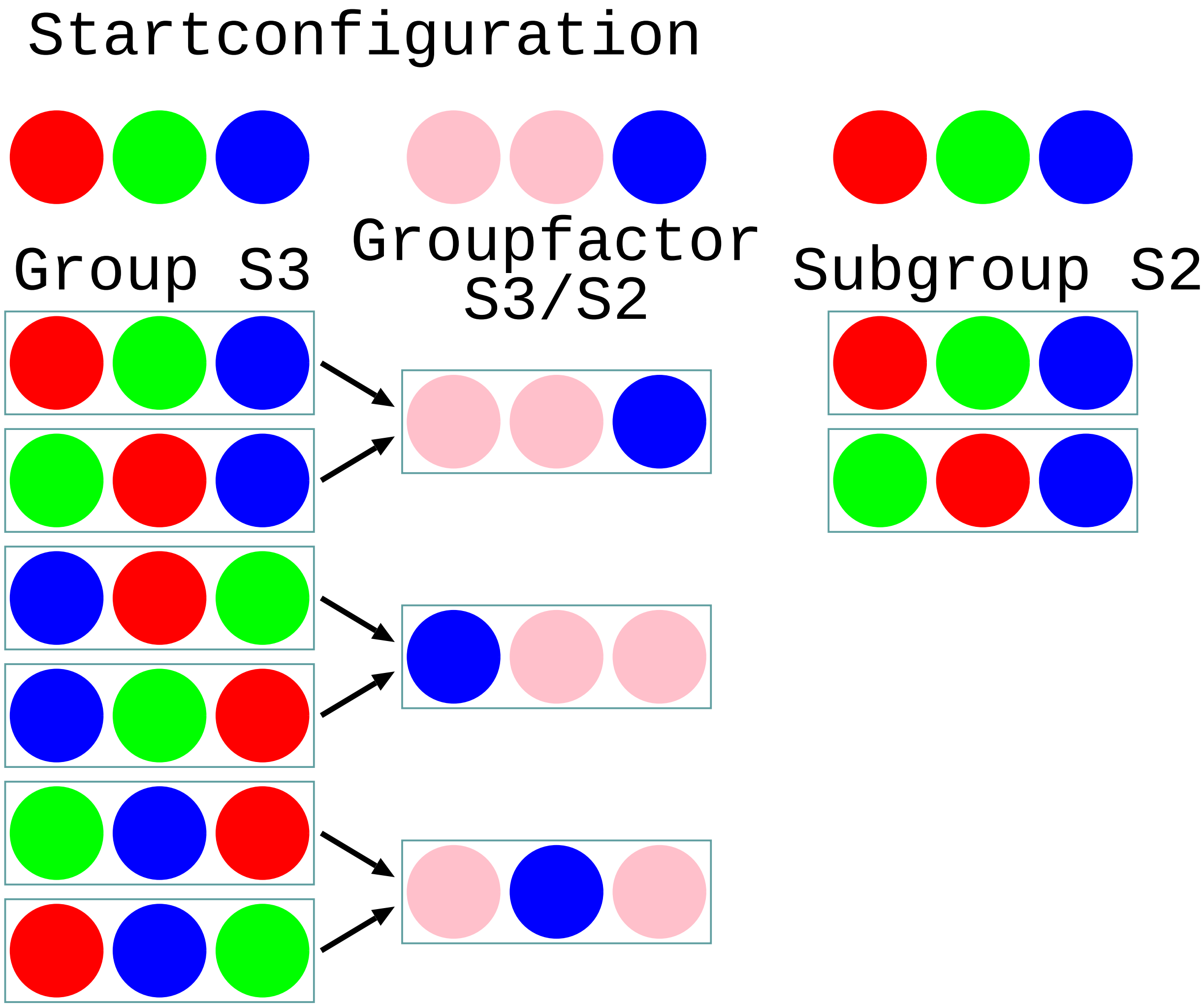
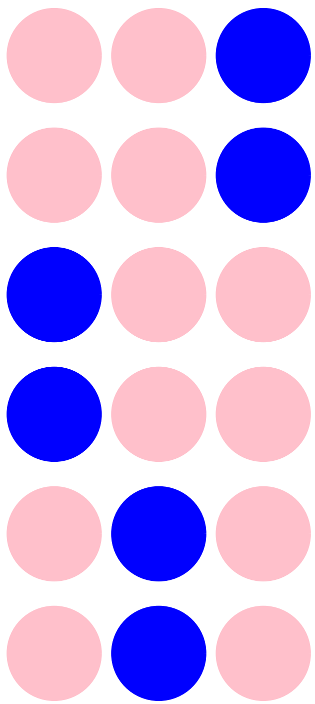


Application to example

- **Exchangebility of nondisabled workers**



S3/S2	$\dot{T}_0(g)$	$\dot{T}_1(g)$	$\dot{T}_2(g)$
	(1)	$()$	$\begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
	(1)	$()$	$\begin{pmatrix} -\sin 0 & \cos 0 \\ \cos 0 & \sin 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
	(1)	$()$	$\begin{pmatrix} \cos 120 & -\sin 120 \\ \sin 120 & \cos 120 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
	(1)	$()$	$\begin{pmatrix} -\sin 120 & \cos 120 \\ \cos 120 & \sin 120 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
	(1)	$()$	$\begin{pmatrix} \cos 240 & -\sin 240 \\ \sin 240 & \cos 240 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
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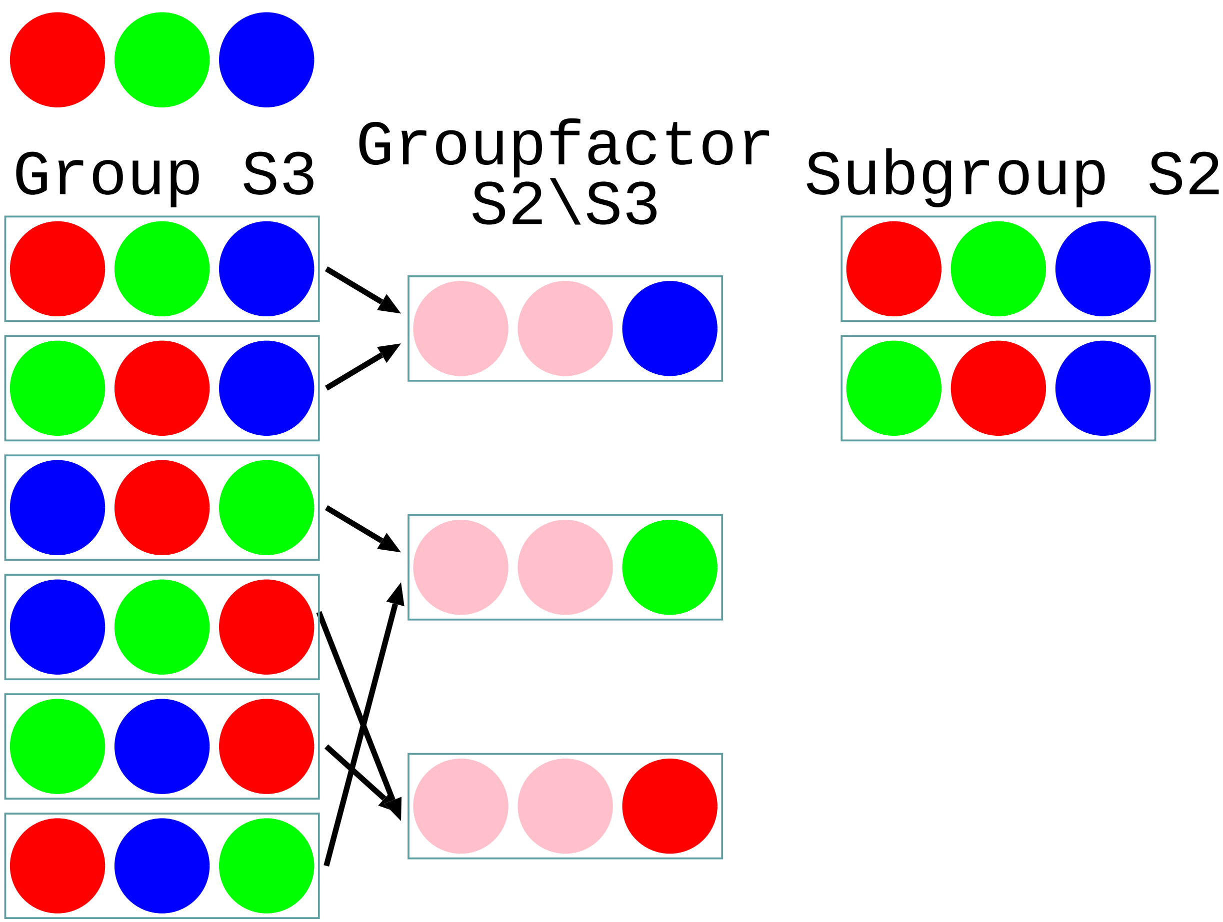
Application to example

- **Exchangebility of nondisabled workers**
- H_1 : **Exchangebility of day shifts and nondisabled workers**

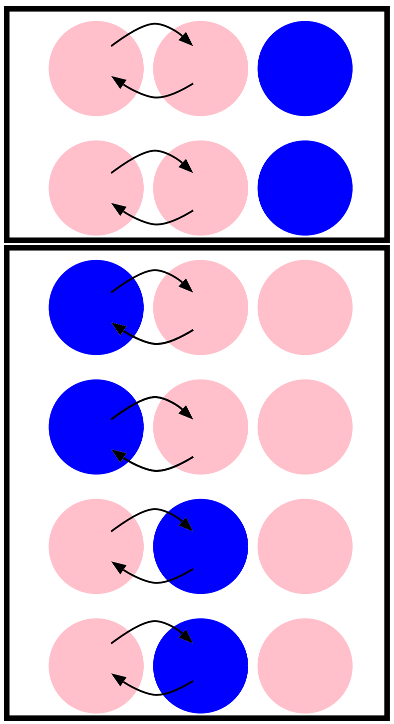
  `Startconfiguration`



Application to example

- **Exchangebility of nondisabled workers**
- H_1 : **Exchangebility of day shifts and nondisabled workers**

S2\S3/S2

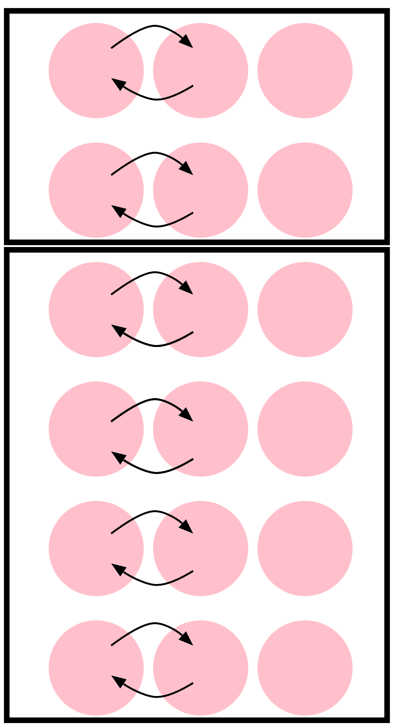


$T_0(g)$	$T_1(g)$	$T_2(g)$
(1)	( )	$(1\ 1)\begin{pmatrix}\cos 0 & -\sin 0\\ \sin 0 & \cos 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=2$
(1)	( )	$(1\ 1)\begin{pmatrix}-\sin 0 & \cos 0\\ \cos 0 & \sin 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=2$
(1)	( )	$(1\ 1)\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=-1$
(1)	( )	$(1\ 1)\begin{pmatrix}-\sin 120 & \cos 120\\ \cos 120 & \sin 120\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=-1$
(1)	( )	$(1\ 1)\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=-1$
(1)	( )	$(1\ 1)\begin{pmatrix}-\sin 240 & \cos 240\\ \cos 240 & \sin 240\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=-1$

Application to example

- **Exchangebility of nondisabled workers**
- H_1 : **Exchangebility of day shifts and nondisabled workers**
- H_0 : **Exchangebility of all three Workers**

S2\S3/S3



$T_0(g)$	$T_1(g)$	$T_2(g)$
(1)	( )	$(1\ 1)\begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}(\ ) = (\ )$
(1)	( )	$(1\ 1)\begin{pmatrix} -\sin 0 & \cos 0 \\ \cos 0 & \sin 0 \end{pmatrix}(\ ) = (\ )$
(1)	( )	$(1\ 1)\begin{pmatrix} \cos 120 & -\sin 120 \\ \sin 120 & \cos 120 \end{pmatrix}(\ ) = (\ )$
(1)	( )	$(1\ 1)\begin{pmatrix} -\sin 120 & \cos 120 \\ \cos 120 & \sin 120 \end{pmatrix}(\ ) = (\ )$
(1)	( )	$(1\ 1)\begin{pmatrix} \cos 240 & -\sin 240 \\ \sin 240 & \cos 240 \end{pmatrix}(\ ) = (\ )$
(1)	( )	$(1\ 1)\begin{pmatrix} -\sin 240 & \cos 240 \\ \cos 240 & \sin 240 \end{pmatrix}(\ ) = (\ )$

Application to example

- **Exchangebility of nondisabled workers**
- H_1 : **Exchangebility of day shifts and nondisabled workers**
- H_0 : **Exchangebility of all three Workers**
- **Our Testproblem is:**

$$\mathbf{H}_0 : E[\overset{\therefore}{T}_2(g)] = 0 \quad \text{vs.} \quad \mathbf{H}_1 : E[\overset{\therefore}{T}_2(g)] \neq 0$$

With

$$\overset{\therefore}{T}_2(g) = \begin{cases} 2, & \text{if Mr. Blue is in night shift} \\ -1, & \text{if Mr. Blue is in day shift} \end{cases}$$

Application to example

- **Exchangebility of nondisabled workers**
- H_1 : **Exchangebility of day shifts and nondisabled workers**
- H_0 : **Exchangebility of all three Workers**
- **Our Testproblem is:**

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With

$$\overset{\therefore}{T}_2(g) = \begin{cases} 2, & \text{if Mr. Blue is in night shift} \\ -1, & \text{if Mr. Blue is in day shift} \end{cases}$$

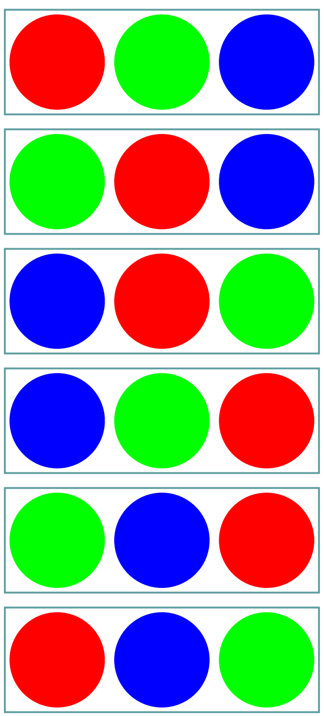
Maybe this is not so surprising, however this was the most simple example.

Is Mr. Blue discriminated?

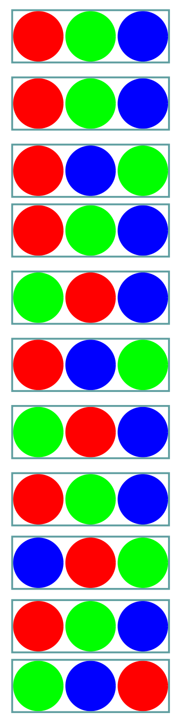
$$\mathbf{mean}\overset{\therefore}{T}_2 = \frac{1}{11} \sum_{i=1}^{11} \overset{\therefore}{T}_2(g_i) = 0.9090909$$

The statistical data

possible outcomes:



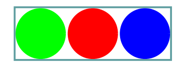
Each week the boss assigns the workers to shifts:

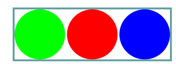
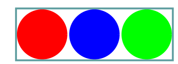


Is Mr. Blue discriminated?

- The test problem

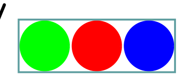
$\mathbf{H}_0: \quad P(A) = P(\sigma A \tau),$

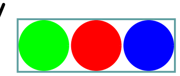
for all $\sigma \in \mathbf{span}\ \langle$  $\rangle$,

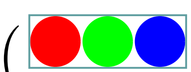
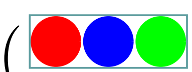
and for all $\tau \in \langle$ ,  $\rangle$

vs.

$\mathbf{H}_1: \quad P(A) = P(\sigma A \tau),$

for all $\sigma \in \mathbf{span}\ \langle$  $\rangle$,

and for all $\tau \in \langle$  $\rangle$,

and $P($  $) > P($  $)$

Is Mr. Blue discriminated?

- The test problem

$$\mathbf{H}_0 : \quad E[\overset{\therefore}{T}_2(g)] = 0, \; P = \text{Uniform}$$

vs.

$$\mathbf{H}_1 : \quad E[\overset{\therefore}{T}_2(g)] > 0$$

Is Mr. Blue discriminated?

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- exact p-value = 0.03863

Is Mr. Blue discriminated?

- The test problem

$$\mathbf{H}_0: \quad E[\overset{\therefore}{T}_2(g)] = 0, \; P = \text{Uniform}$$

vs.

$$\mathbf{H}_1: \quad E[\overset{\therefore}{T}_2(g)] > 0$$

- exact p-value = 0.03863
- **Result:** Mr. Blue gets significantly more night shifts than his comparable colleagues Mr. Green and Mr. Red.

Elements of group scale statistics

- **Moments / characteristic transform**

$$\mu_l = E[T_l]$$

Elements of group scale statistics

- **Moments / characteristic transform**
- **Testproblems based on symmetry**

$$\mathbf{H}_0 : N_l^S \mu_l(g) = 0 \text{ vs. } \mathbf{H}_1 : N_l^S \mu_l(g) \neq 0$$

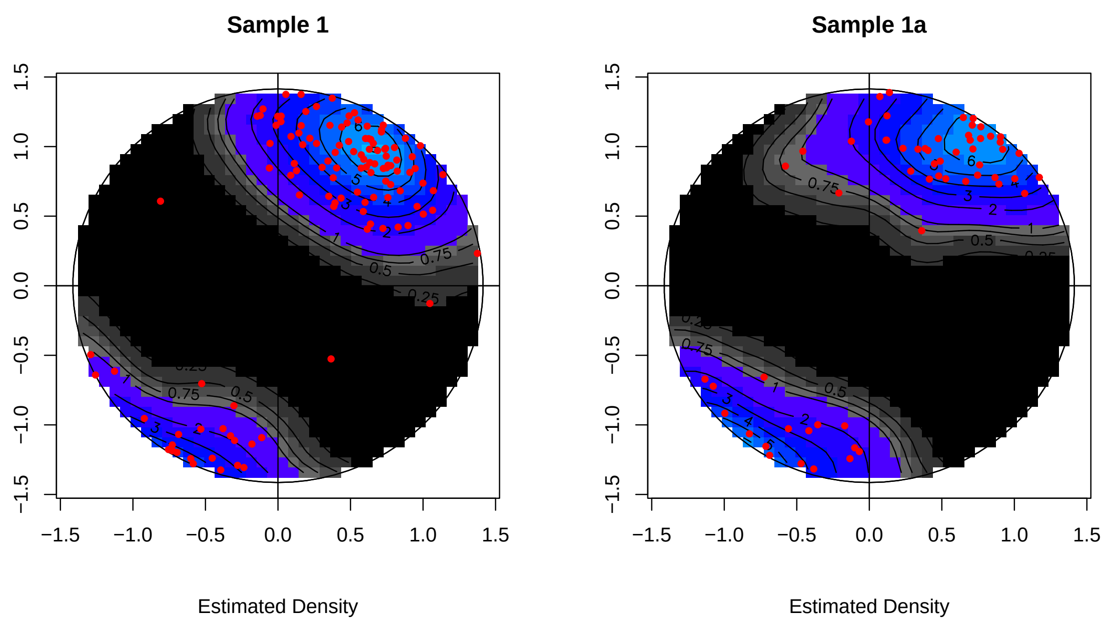
Elements of group scale statistics

- **Moments / characteristic transform**
- **Testproblems based on symmetry**
- **Distances**
  based on

$$\|T_l(g_1) - T_l(g_2)\|$$

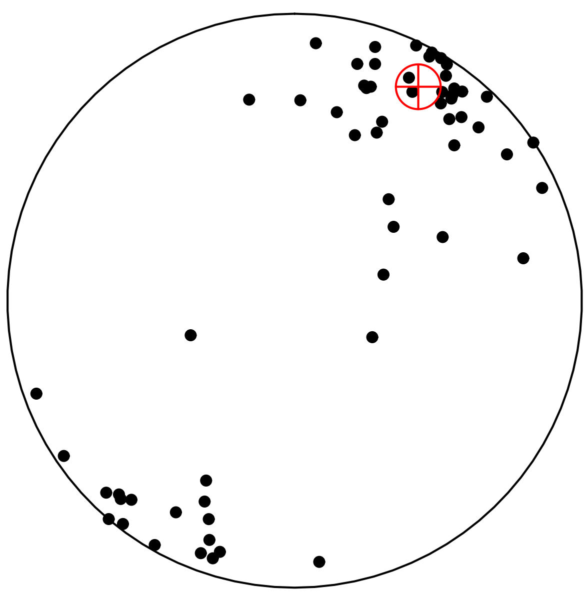
Elements of group scale statistics

- **Moments / characteristic transform**
- **Testproblems based on symmetry**
- **Distances**
- **Kernel density estimation**
  based on group convolution, moments and distance



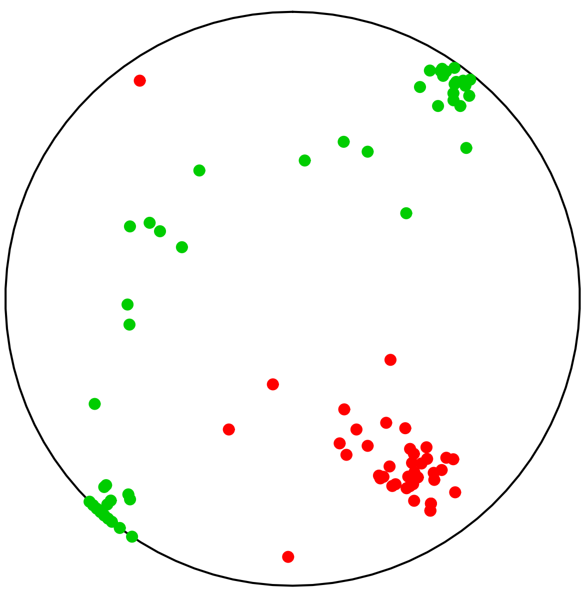
Elements of group scale statistics

- **Moments / characteristic transform**
- **Testproblems based on symmetry**
- **Distances**
- **Kernel density estimation**
- **Location and spread parameters**
  based on distances



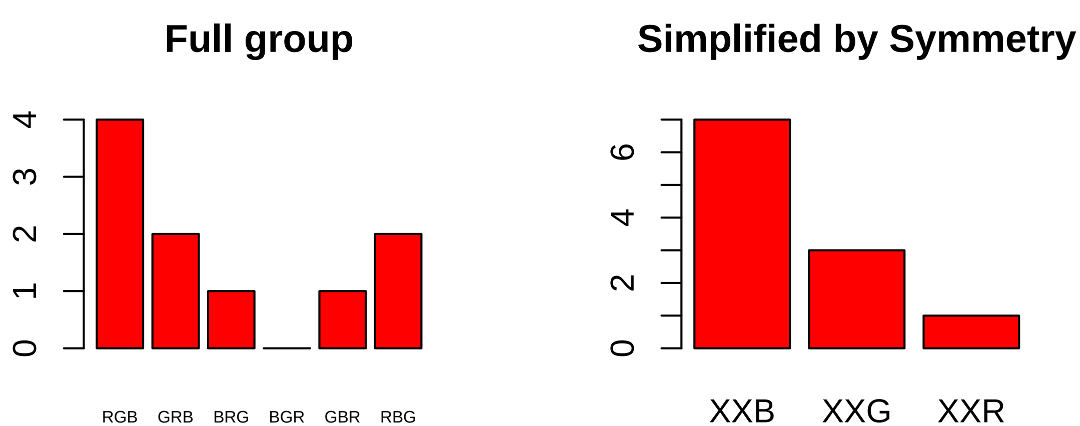
Elements of group scale statistics

- **Moments / characteristic transform**
- **Testproblems based on symmetry**
- **Distances**
- **Kernel density estimation**
- **Location and spread parameters**
- **Clusteranalysis**
  based on distances



Elements of group scale statistics

- **Moments / characteristic transform**
- **Testproblems based on symmetry**
- **Distances**
- **Kernel density estimation**
- **Location and spread parameters**
- **Clusteranalysis**
- **Graphics**
  based on group quotients and representations



Elements of group scale statistics

- **Moments / characteristic transform**
- **Testproblems based on symmetry**
- **Distances**
- **Kernel density estimation**
- **Location and spread parameters**
- **Clusteranalysis**
- **Graphics**
- **Symmetric Beran type exponential families**
  based on Haar measure and $T_l(g)$ as sufficient statistics

$$\frac{dP}{d\eta}(g) = A(\theta) \exp\left(\sum_{l=1}^{L} \mathbf{tr}\, \theta_l^t \, \overset{\therefore}{T}_l(g) \right)$$

Elements of group scale statistics

- **Moments / characteristic transform**
- **Testproblems based on symmetry**
- **Distances**
- **Kernel density estimation**
- **Location and spread parameters**
- **Clusteranalysis**
- **Graphics**
- **Symmetric Beran type exponential families**
- **Dependence models / group regression**
  based on generalised linear models and exponential families. e.g.

$$\theta_1 = M + \alpha T_1(x)$$

Conclusions

- **Group scale is substantially different from real and categorial scale.**
  No $+$, No 0, No $<$, No $\mathbb{R}^d$

Conclusions

- **Group scale is substantially different from real and categorial scale.**
- **Group scale has its own statistical questions.**
  . . . symmetry, symmetry, symmetry

Conclusions

- **Group scale is substantially different from real and categorial scale.**
- **Group scale has its own statistical questions.**
- **Group scale has its own statistical methods and models.**
  . . . easy to explain in 25 hours

Conclusions

- **Group scale is substantially different from real and categorial scale.**
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- **Group scale has its own statistical methods and models.**
- **Group scale comes in many guises and always as special case.**
  . . . needing to be recognized.

Conclusions

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- **Group scale has its own statistical methods and models.**
- **Group scale comes in many guises and always as special case.**
- **Methods for group scale are – also desirable – not in generality available.**

  . . . especially not in software.

Conclusions

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Thank you for your attention